J. DIFFERENTIAL GEOMETRY
20 (1984) 23–53

A DUALITY THEOREM FOR WILLMORE SURFACES

ROBERT L. BRYANT

0. Introduction

In 1965 T. J. Willmore [12] proposed to study the functional

$$\tilde{\mathscr{W}}(X) = \int_M H^2\, dA$$

on immersions X: $M^2 \to \mathbf{E}^3$, where M^2 is a compact surface, H is the mean curvature of the immersion, and dA is the induced area from (or area density if M is not oriented). If we define

$$\mathscr{W}(X) = \int_M (H^2 - K)\, dA,$$

then by the Gauss-Bonnet theorem

$$\tilde{\mathscr{W}}(X) = \mathscr{W}(X) + 2\pi\chi(M),$$

so the two functionals differ by a constant. The functional $\mathscr{W}(X)$ has the advantage that its integrand is nonnegative and vanishes exactly at the umbilic points of the immersion X.

Obviously $\mathscr{W}(X) = 0$ iff $M^2 = S^2$ and X is totally umbilic. Thus, the absolute minimum of $\mathscr{W}$ on the space of immersions X: $S^2 \to \mathbf{E}^3$ is 0 and the critical locus of such X is known. When M is a torus, Willmore provided an example of an immersion X: $M \to \mathbf{E}^3$ with $\mathscr{W}(X) = 2\pi^2$ and showed that $\mathscr{W}(X) \geqslant 2\pi^2$ for all smooth surfaces of revolution. He then conjectured that $\mathscr{W}(X) \geqslant 2\pi^2$ for all immersions of the torus with equality only for the example he provided: the anchor ring swept out by revolving a circle of radius r about the line whose distance from the center of the circle was $r\sqrt{2}$. White then pointed out that the two-form $(H^2 - K)\, dA$ had the property of being invariant under conformal transformations of $\mathbf{E}^3$ plus the "point at infinity"

Received August 17, 1983 and, in revised form, June 4, 1984. This research was done while the author was an Alfred P. Sloan Fellow.

and hence that the cyclides of Dupin generated from Willmore's anchor ring by conformal transformations must also satisfy $\mathscr{W}(X) = 2\pi^2$. The conjecture was then modified so that equality was supposed to hold only if the immersion was conformally equivalent to Willmore's anchor ring.

In 1982, Li and Yau introduced the notation of conformal area, $V_c(M)$, for a surface M with a fixed conformal structure. They then showed that, for a conformal immersion X: $M^2 \to \mathbf{E}^3$,

$$\tilde{\mathscr{W}}(X) \geqslant V_c(M).$$

Since they were able to show that $V_c(M) \geqslant 2\pi^2$ for M a torus with a conformal structure near that of the square torus, this proves part of Willmore's conjecture.

In this paper, we study the Willmore functional using the conformal invariance from the outset. In §1 we develop the structure equations for conformal three-space (i.e., S^3). We then apply the moving frame to study immersed surfaces X: $M^2 \to S^3$. We define a conformally invariant 2-form Ω_X on M and show that for any stereographic projection ρ: $S^3 - \{y_0\} \to \mathbf{E}^3$, the equation

$$\Omega_X = (H^2 - K)\, dA$$

holds, where the quantities on the right are computed for the immersion $\rho \circ X$: $M \to \mathbf{E}^3$. This demonstrates the conformal invariance of the Willmore integrand and the conformal invariance of the umbilic locus $\mathscr{U}_X = \{m \in M | \Omega_X(m) = 0\}$.

We then construct, on the compliment of the umbilic locus, a smooth map $\hat{X}$: $M - \mathscr{U}_X \to S^3$ with the defining property that if $m_0 \notin \mathscr{U}_X$, then $\hat{X}(m_0)$ is the unique point so that the mean curvature of $\rho \circ X$ vanishes to second order at m_0 for any stereographic projection ρ: $S^3 - \{\hat{X}(m_0)\} \to \mathbf{E}^3$. We call $\hat{X}$ the *conformal transform* of X. Unfortunately, it is not true, in general, that $\hat{\hat{X}} = X$.

In §2, we compute the Euler-Lagrange equation for the functional $\mathscr{W}$. In Euclidean terms, this is known to be the equation

$$\Delta H + 2(H^2 - K)H = 0.$$

Our derivation is conformally invariant and leads us to consider the complex structure on M^2 induced by the induced conformal structure and a choice of orientation on M^2. We say that an immersion X: $M^2 \to S^3$ is a *Willmore immersion* if it is a critical point of the Willmore functional. In §3 we prove two basic theorems relating the Willmore immersion to the complex structure. The first, Theorem B, constructs a holomorphic quartic differential on M from the Willmore immersion X, denoted $\mathscr{Q}_X$. The second, Theorem C, shows that if a Willmore immersion is not totally umbilic, then the conformal transform completes smoothly to a branched conformal immersion $\hat{X}$: $M \to S^3$. If $\mathcal{Q}_X \equiv 0$, then $\hat{X}$ is constant. If $\mathcal{Q}_X \not\equiv 0$, then $\hat{X}$ is also a Willmore (branched) immersion and satisfies $\hat{\hat{X}} = X$. We say that $\hat{X}$ is the *Willmore dual* of X.

In §4, we use the fact that $\mathcal{Q}_X \equiv 0$ for $M = S^2$ to completely classify the Willmore immersions X: $S^2 \to S^3$ in terms of a special family of minimal surfaces of finite total curvature in $\mathbf{E}^3$. In turn, we reduce this problem to an algebraic geometry problem concerning zeros and poles of meromorphic functions on $\mathbf{CP}^1$. In fact we show that all the critical values of $\mathscr{W}$ on spherical immersions are nonnegative multiples of 4π. This result is closely related to another theorem of Li and Yau. They show that for any immersion of a compact surface X: $M^2 \to \mathbf{E}^3$ the inequality $\tilde{\mathscr{W}}(X) \geqslant 4\pi k$ holds, where k is the maximum number of points in $X^{-1}(p)$ as p ranges over $\mathbf{E}^3$. We show that, for Willmore immersions satisfying $\mathcal{Q}_X \equiv 0$, equality always holds in their theorem. It is an interesting question whether or not equality implies $\mathcal{Q}_X \equiv 0$.

Finally in §5, we compute an example. We find there is a 4-parameter family of Willmore immersions X: $S^2 \to S^3$ with $\mathscr{W}(X) \equiv 12\pi$ (the next nontrivial case after $\mathscr{W}(X) = 0$). This parameter space is noncompact and its members are inequivalent under reparametrizations in S^2 and conformal transformations in S^3. We indicate that the moduli space of Willmore immersions with $\mathscr{W}(X) = 4(2d+1)\pi$ is of dimension $4d$ for $d \geqslant 0$. We then close with a short discussion of the cases where X: $M^2 \to S^3$ is a *branched* conformal Willmore immersion.

The methods of the moving frame are used throughout the paper. The basic reference on Riemann surfaces has been [5] and our notations for divisors and line bundles are consistent with this reference. It is a pleasure to thank Phillip Griffiths for the many interesting discussions concerning Riemann surfaces and invariant variational problems which inspired this work.

1. Conformal geometry of surfaces in S^3

We first describe the standard model of S^3 as a conformal space. On $\mathbf{R}^5$, we consider the standard Minkowski inner product

$$\langle x, y\rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4,$$

where $x = (x^a)$, $y = (y^a)$ and we use the index range $0 \leqslant a, b, c \leqslant 4$. Following the terminology of relativity, we say that an $x \in \mathbf{R}^5$ is *space-like* if $\langle x, x\rangle > 0$, *time-like* if $\langle x, x\rangle < 0$, and *light-like* (or *null*) if $x \neq 0$ but $\langle x, x\rangle = 0$. We fix an orientation on $\mathbf{R}^5$ by requiring $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$

$\wedge\, dx^4 > 0$, and we fix a time-orientation on $\mathbf{R}^5$ by saying that a time-like or light-like vector $x \in \mathbf{R}^5$ is *positive* (or *future directed*) if $x^0 > 0$. For brevity, we will simply write $\mathbf{L}^5$ to denote $\mathbf{R}^5$ together with the three choices of inner product, orientation, and time-orientation.

The space of null lines through $0 \in \mathbf{L}^5$ forms a smooth manifold diffeomorphic to the three-sphere and henceforth will be denoted by S^3. Let $\mathscr{L}^+$ denote the space of positive null vectors in $\mathbf{L}^5$. For each $x \in \mathscr{L}^+$, we denote the line spanned by x by (x). This gives us a map $\mathscr{L}^+ \to S^3$ which is a smooth submersion. The fibers of this map are the positive null rays emanating from $0 \in \mathbf{L}^5$. The metric on $\mathbf{L}^5$ restricts to the hypersurface $\mathscr{L}^+$ to be a degenerate inner product of type $(3,0)$. The null space at each $x \in \mathscr{L}^+$ is the tangent to the line (x). The dilation $x \mapsto rx$ $(r > 0)$ multiplies this metric by a factor of r^2. It follows that, up to a positive factor, this metric descends to S^3. In this way, S^3 inherits a natural conformal structure. If v_1, v_2, v_3 form a basis of $T_{(x)}S^3$, let $e_1 e_2, e_3 \in T_x\mathscr{L}^+$ denote a set of preimages under the map $T_x\mathscr{L}^+ \to T_{(x)}S^3$. We say that $v_1 \wedge v_2 \wedge v_3 > 0$ if $x \wedge e_1 \wedge e_2 \wedge e_3 \wedge y > 0$, where $y \in \mathscr{L}^+$ but $y \notin (x)$. It is an exercise to check that this well-defines an orientation on S^3 (independent of our choices of x, e, and y).

An *automorphism* of $\mathbf{L}^5$ is a linear automorphism of $\mathbf{R}^5$ which preserves $\langle\ ,\ \rangle$, the orientation and the time-orientation. We denote the group of automorphisms of $\mathbf{L}^5$, by $\mathrm{Aut}(\mathbf{L}^5)$. This is known to be a connected Lie group of dimension 10 and is isomorphic to the identity component of the group $\mathrm{SO}(4,1)$, see [7]. The group $\mathrm{Aut}(\mathbf{L}^5)$ acts on S^3 in the obvious way and induces a group of conformal, orientation-preserving diffeomorphisms of S^3. It is a classical theorem that each conformal, orientation-preserving diffeomorphism of S^3 is induced by a unique element of $\mathrm{Aut}(\mathbf{L}^5)$, see [1].

Another classical model of S^3 is "$\mathbf{E}^3$ with a point at infinity". Because we will need to compare surface theory in $\mathbf{E}^3$ with conformal surface theory in S^3, we comment on how this transition is made. First, we note that, for $x, y \in \mathscr{L}^+$, we have $\langle x, y\rangle \leqslant 0$ with $\langle x, y\rangle = 0$ iff $x \wedge y = 0$. Thus, if we set

$$E_y = \{x \in \mathscr{L}^+ | \langle x, y\rangle = -1\},$$

then the natural map $E_y \hookrightarrow \mathscr{L}^+ \to S^3$ establishes a diffeomorphism $E_y \xrightarrow{\sim} S^3 - \{(y)\}$. If we give E_y the (positive definite) metric induced on it as a submanifold of $\mathbf{L}^5$, then this map is obviously conformal.

Proposition 1. *The space E_y with its induced metric is isometric to $\mathbf{E}^3$.*

Proof. Let $x_0 \in E_y$ be fixed and define the affine map $P\colon \mathbf{L}^5 \to (x_0)^\perp$

$$P(z) = z - x_0 + \langle x_0, z\rangle y.$$

This map establishes a diffeomorphism between E_y and the three-plane $(x_0, y)^\perp$ (necessarily a space-like plane). The inverse is give by the quadratic map

$$Q(w) = x_0 + w + \tfrac{1}{2}\langle w, w\rangle y, \qquad w \in (x_0, y)^\perp .$$

It remains to show that this is an isometry if we give $(x_0, y)^\perp$ the induced metric. However, if $z \in E_y$ and $\dot z \in T_z E_y$, then we have $\langle \dot z, y\rangle = 0$. We compute

$$\begin{aligned}\langle P_*(\dot z), P_*(\dot z)\rangle &= \langle \dot z + \langle x_0, \dot z\rangle y, \dot z + \langle x_0, \dot z\rangle y\rangle \\ &= \langle \dot z, \dot z\rangle + 2\langle x_0, \dot z\rangle\langle \dot z, y\rangle + \langle x_0, \dot z\rangle^2\langle y, y\rangle \\ &= \langle \dot z, \dot z\rangle. \quad \text{q.e.d.}\end{aligned}$$

In order to study surface theory in S^3, we introduce moving frames. Let $B = (B_{ab})$ denote the symmetric matrix

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_3 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We let $\mathscr{F}$ denote the space of positively oriented bases $\mathscr{E} = (e_0, e_1, e_2, e_3, e_4) = (e_a)$ of $\mathbf{L}^5$ satisfying the condition

$$\langle e_a, e_b\rangle = B_{ab}, \qquad e_0, e_4 \in \mathscr{L}^+.$$

The group Aut($\mathbf{L}^5$) acts simply transitively on $\mathscr{F}$ in the obvious way. This shows that $\mathscr{F}$ is a connected smooth manifold of dimension 10.

We let

$$O(B) = \left\{ g \in M_{5\times 5}(\mathbf{R}) \,|\, {}^t g B g = B \right\},$$

and we let G be the connected component of the identity $I_5 \in O(B)$. Note that G is isomorphic to Aut($\mathbf{L}^5$), though not canonically. G acts naturally on the right of $\mathscr{F}$ by the formula

$$\mathscr{E} \cdot g = (e_a) \cdot g = \left(e_b g_a^b\right),$$

where $g = (g_a^b) \in G$. This action is also simply transitive. Hence we may identify $\mathscr{F}$ with G up to a left translation in G.

If we regard the components of $\mathscr{E} \in \mathscr{F}$ as determining $\mathbf{L}^5$-valued functions on $\mathscr{F}$, e_a: $\mathscr{F} \to \mathbf{L}^5$, then we may compute their exterior derivatives, de_a as vector-valued 1-forms on $\mathscr{F}$. Since the e_a form a basis, there exist unique 1-forms ω_b^a on $\mathscr{F}$ satisfying

(1.1) $$de_a = e_b \omega_b^a.$$

Differentiating (1.1), we get

(1.2) $$d\omega_b^a = -\omega_c^a \wedge \omega_b^c.$$

These are the *structure equations* of É. Cartan. If we differentiate the relation $\langle e_a, e_b \rangle = B_{ab}$ and set $\omega_{ab} = B_{ac}\omega^c_b$, then we get the relations

$$\omega_{ab} + \omega_{ba} = 0. \tag{1.3}$$

Thus, only ten of the ω^a_b are independent. Under an identification of $\mathscr{F}$ with G up to left translation, these are a basis for the left-invariant forms on G.

The map $(e_0)\colon \mathscr{F} \to S^3$ makes $\mathscr{F}$ into a fiber bundle over S^3 with fiber

$$G_0 = \{ g \in G \mid e_0(\ell \cdot g) = e_0(\ell) \text{ for all } \ell \in \mathscr{F} \}.$$

It is easy to compute that

$$G_0 = \left\{ \begin{pmatrix} r^{-1} & {}^t cA & \frac{1}{2} r\, {}^t cc \\ 0 & A & rc \\ 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} r > 0,\ c \in M_{3\times 1}(\mathbf{R}) \\ \text{and } A \in \mathrm{SO}(3) \subseteq M_{3\times 3}(\mathbf{R}) \end{array} \right\}.$$

From this, we see that the forms $\{\omega^1_0, \omega^2_0, \omega^3_0\}$ span the semibasic forms for the projection $(e_0)\colon \mathscr{F} \to S^3$. In fact, the symmetric quadratic form $(\omega^1_0)^2 + (\omega^2_0)^2 + (\omega^3_0)^2$ and the exterior 3-form $\omega^1_0 \wedge \omega^2_0 \wedge \omega^3_0$ are well defined up to a positive multiple on S^3 and determine the conformal structure and orientation respectively.

Now let M^2 be an oriented surface and suppose we are given a smooth immersion $X\colon M^2 \to S^3$. We want to study its geometric invariants under the conformal group of S^3.

We define the 0th order frame bundle of X, $\mathscr{F}^{(0)}_X$, by

$$\mathscr{F}^{(0)}_X = \{ (p, \ell) \in M \times \mathscr{F} \mid X(p) = (e_0)(\ell) \}.$$

We have a diagram:

$$\begin{array}{ccc} \mathscr{F}^{(0)}_X & \xrightarrow{\ \ell\ } & \mathscr{F} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle (e_0)} \\ M & \xrightarrow{\ X\ } & S^3 \end{array}$$

We will now work on $\mathscr{F}^{(0)}_X$. Following the usual practice in the theory of moving frames (see [2]) we will write ϕ instead of $\ell^*(\phi)$ to denote forms on $\mathscr{F}^{(0)}_X$ which are pulled back from $\mathscr{F}$. This should cause no confusion as long as we clearly specify the manifold on which we are working.

The forms $\{\omega^1_0, \omega^2_0, \omega^3_0\}$ on $\mathscr{F}^{(0)}_X$ are now semibasic for p. Since M has dimension 2, there must be a relation among these three forms and because $p\colon \mathscr{F}^{(0)}_X \to M$ is a submersion, there cannot be more than one such linear relation.

If $g \in G_0$ is of the form

$$g = \begin{pmatrix} r^{-1} & {}^t cA & \frac{1}{2} r\,{}^t cc \\ 0 & A & rc \\ 0 & 0 & r \end{pmatrix},$$

then one computes that

$$R_g^* \begin{pmatrix} \omega_0^1 \\ \omega_0^2 \\ \omega_0^3 \end{pmatrix} = r^{-1} A^{-1} \begin{pmatrix} \omega_0^1 \\ \omega_0^2 \\ \omega_0^3 \end{pmatrix},$$

where $R_g\colon \mathscr{F}_X^{(0)} \to \mathscr{F}_X^{(0)}$ is $R_g(p, \ell) = (p, \ell \cdot g)$.

It follows that we may define the first-order frame bundle, $\mathscr{F}_X^{(1)}$, by

$$\mathscr{F}_X^{(1)} = \left\{ (p, \ell) \in \mathscr{F}_X^{(0)} \middle| \omega^3_{0|(p,\ell)} = 0, \left(\omega_0^1 \wedge \omega_0^2\right)_{|(p,\ell)} > 0 \right\}.$$

We remark that, because M^2 is assumed oriented, the 2-forms on M which do not vanish are divided into positive and negative; so too are the semibasic, nonvanishing 2-forms on $\mathscr{F}_X^{(0)}$. Thus, once we impose the condition $\omega_0^3 = 0$, the nonvanishing of $\omega_0^1 \wedge \omega_0^2$ allows us to choose its sign. We define the group

$$G_1 = \left\{ \begin{pmatrix} r^{-1} & {}^t pA & \frac{1}{2} r\,{}^t pp \\ 0 & A & rp \\ 0 & 0 & r \end{pmatrix} \in G_0 \,\middle|\, A = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, c^2 + s^2 = 1 \right\}$$

and we note that $p\colon \mathscr{F}_X^{(1)} \to M$ is a right principal G_1-bundle over $\mathscr{F}_X^{(1)}$. We now restrict the forms on $\mathscr{F}_X^{(0)}$ to $\mathscr{F}_X^{(1)}$. The forms $\{\omega_0^1, \omega_0^2\}$ are now a basis for the semibasic forms. Because $\omega_0^3 = 0$ and because $\omega_0^4 = 0$ by (1.3), we compute

$$0 = d\omega_0^3 = -\omega_1^3 \wedge \omega_0^1 - \omega_2^3 \wedge \omega_0^2.$$

It follows, by Cartan's Lemma, that there exist smooth functions $h_{ij} = h_{ji}$ on $\mathscr{F}_X^{(1)}$ so that $\omega_i^3 = h_{ij}\omega_0^j$. Here, we adopt the index range $1 \leqslant i, j \leqslant 2$.

One now computes that

$$R_g^* \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = r \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} h_{11} - p^3 & h_{12} \\ h_{21} & h_{22} - p^3 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

when

$$g = \begin{pmatrix} r^{-1} & {}^t pA & \frac{1}{2} r\,{}^t pp \\ 0 & A & rp \\ 0 & 0 & r \end{pmatrix} \in G_1$$

with

$$A = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad p = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}.$$

Correspondingly, if we differentiate the equations $\omega_i^3 = h_{ij}\omega_0^j$ and apply Cartan's Lemma, we get the infinitesimal version of the above equation:

$$dh_{ij} = -\delta_{ij}\omega_3^0 + h_{ij}\omega_0^0 + h_{ik}\omega_j^k + h_{ki}\omega_i^k + h_{ijk}\omega_0^k,$$

where $\{h_{ijk}\}$ are smooth functions on $\mathscr{F}_X^{(1)}$ symmetric in all indices ($1 \leqslant i, j, k \leqslant 2$).

It follows that, *without making any further nondegeneracy assumptions*, we can always make a *partial* second-order reduction:

$$\mathscr{F}_X^{(\gamma)} = \left\{ (p, \ell) \in \mathscr{F}_X^{(1)} | (h_{11} + h_{22})(p, \ell) = 0 \right\} \subseteq \mathscr{F}_X^{(1)}.$$

This is a G_γ-bundle over M, where

$$G_\gamma = \left\{ \begin{pmatrix} r^{-1} & {}^t pA & \frac{1}{2} r\, {}^t pp \\ 0 & A & rp \\ 0 & 0 & r \end{pmatrix} \in G_1 \middle| p = \begin{pmatrix} p^1 \\ p^2 \\ 0 \end{pmatrix} \right\}.$$

Our formulae imply that, on $\mathscr{F}_X^{(1)}$, for $g \in G_1$,

$$R_g^*\left(\tfrac{1}{4}(h_{11} - h_{22})^2 + h_{12}^2\right) = r^2\left(\tfrac{1}{4}(h_{11} - h_{22})^2 + h_{12}^2\right),$$

$$R_g^*\left(\omega_0^1 \wedge \omega_0^2\right) = r^{-2}\omega_0^1 \wedge \omega_0^2.$$

Thus, there exists a smooth 2-form on M, Ω_X, which satisfies

$$p^*(\Omega_X) = \left(\tfrac{1}{4}(h_{11} - h_{22})^2 + h_{12}^2\right)\omega_0^1 \wedge \omega_0^2.$$

Note that $\Omega_X \geqslant 0$. We define the *umbilic locus* of X by $\mathscr{U}_X = \{p \in M | (\Omega_X)_p = 0\}$. This terminology will be justified below. Note that $\mathscr{U}_X$ is closed. Let $\mathscr{N}_X \subseteq M$ denote the compliment of $\mathscr{U}_X$ in M. We assume that $\mathscr{N}_X \neq \varnothing$. We can define the second order frame bundle, $\mathscr{F}_X^{(2)}$, over $\mathscr{N}_X$ by

$$\mathscr{F}_X^{(2)} = \left\{ (p, \ell) \in \mathscr{F}_X^{(1)} \middle| \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

This is a G_2-bundle over $\mathscr{N}_X$, where

$$G_2 = \left\{ \begin{pmatrix} 1 & {}^t pA & \frac{1}{2} r\, {}^t pp \\ 0 & A & p \\ 0 & 0 & 1 \end{pmatrix} \middle| A = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, c^2 = 1, p = \begin{pmatrix} p^1 \\ p^2 \\ 0 \end{pmatrix} \right\}.$$

Note that $G_2 \subseteq G_\gamma$ and that $\mathscr{F}_X^{(2)} \subseteq \mathscr{F}_X^{(\gamma)}$.

Our formulae for dh_{ij} now implies $\omega_3^0 = h_1\omega_0^1 + h_2\omega_0^2$, where we have set $h_j = \frac{1}{2}(h_{11j} + h_{22j})$ for brevity. We then compute

$$R_g^*\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_1 + p^1 \\ h_2 - p^2 \end{pmatrix}$$

when

$$g = \begin{pmatrix} 1 & cp^1 & cp^2 & 0 & \frac{1}{2}\left((p^1)^2 + (p^2)^2\right) \\ 0 & c & 0 & 0 & p^1 \\ 0 & 0 & c & 0 & p^2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_2.$$

It follows that we may define the third order frame bundle, $\mathscr{F}_X^{(3)}$, over $\mathscr{N}_X$ by

$$\mathscr{F}_X^{(3)} = \left\{(p, \ell) \in \mathscr{F}_X^{(2)} \mid h_1 = h_2 = 0\right\}.$$

Now $G_3 \subseteq G_2$ is a discrete group isomorphic to $\mathbf{Z}/(2)$. It is defined by the above formula with $p^1 = p^2 = 0$.

Note that, for $g \in G_\gamma$, we have the identity $e_3(\ell \cdot g) = e_3(\ell)$. It follows that $e_3\colon \mathscr{F}_X^{(\gamma)} \to \mathbf{L}^5$ is constant on the fibers of the G_γ-bundle $p\colon \mathscr{F}_X^{(\gamma)} \to M^2$ and hence there exists a unique smooth map $\gamma_X\colon M^2 \to \mathbf{L}^5$ so that $e_3 = \gamma_X \circ p = p^*(\gamma_X)$ on $\mathscr{F}_X^{(\gamma)}$.

Note that because $\langle e_3, e_3\rangle \equiv 1$, we see that $\gamma_M\colon M^2 \to Q$, where

$$Q = \left\{x \in \mathbf{L}^5 \mid \langle x, x\rangle = 1\right\}.$$

For reasons that will be made clear below, we call $\gamma_X\colon M^2 \to Q$ the *conformal Gauss map* of the immersion X.

Obviously, $e_4\colon \mathscr{F}_X^{(3)} \to \mathbf{L}^5$ is constant on the fibers of $p\colon \mathscr{F}_X^{(3)} \to \mathbf{L}^5$. Thus we may define a mapping $\hat{X}\colon \mathscr{N}_X \to S^3$ by the formula

$$\hat{X}(p) = \left(e_4(p, \ell)\right), \qquad (p, \ell) \in \mathscr{F}_X^{(3)}.$$

We call $\hat{X}$ the *conformal transform* of the immersion X (when $\mathscr{N}_X \neq \varnothing$).

These two associated maps play an important role in the sequel. We will now pause to interpret them in terms of more familiar classical invariants of Euclidean surface theory.

We have already noted that, by identifying $\mathbf{E}^3$ with $E_y \subseteq \mathscr{L}^+$, we may regard $\mathbf{E}^3$ as S^3 minus a point. Now fix $y \in \mathscr{L}^+$ and let $\mathscr{F}_y = \{\ell = (e_a) \in \mathscr{F} \mid e_4 = y\}$. The projection $e_0\colon \mathscr{F}_y \to E_y$ then makes $\mathscr{F}_y$ into an SO(3)-bundle over E_y. If we restrict the forms ω_b^a to $\mathscr{F}_y$, we get $\omega_4^a = 0$ since $e_a\omega_4^a = de_4 = dy = 0$. Since

(1.3) implies $\omega_4^4 = -\omega_0^0$ we see that on $\mathscr{F}_y$ we have

$$de_0 = e_1\omega_0^1 + e_2\omega_0^2 + e_3\omega_0^3.$$

Thus $\mathscr{F}_y$ may be regarded as the oriented orthonormal frame bundle over E_y. If we are given an immersion $X_0\colon M^2 \to E_y$, we may regard $X(p) = (X_0(p))$ as an immersion $X\colon M^2 \to S^3$. Since we want to compare local surface theory in $\mathbf{E}^3$ with conformal surface theory in S^3, we may suppose that we have chosen a lifting $\ell\colon M^2 \to \mathscr{F}_y$ of X_0 with the property that $\ell(p) = (e_a(p))$ with $e_0(p) = X_0(p)$, with $e_1(p)$ and $e_2(p)$ an oriented tangent basis of $X_{0*}(T_pM)$, and of course, with $e_4(p) \equiv y$. We write $\ell^*(\omega_b^a) = \eta_b^a$ in order to avoid confusion. We then have

$$\eta_4^a = 0, \qquad \eta_0^1 \wedge \eta_0^2 > 0, \qquad \eta_0^3 = 0, \qquad \eta_i^3 = h_{ij}\eta_0^j$$

for some functions h_{ij} on M.

The Gaussian and mean curvatures of the immersion $X_0\colon M^2 \to E_y \simeq \mathbf{E}$ are, respectively

$$K = h_{11}h_{22} - h_{12}^2, \qquad H = \tfrac{1}{2}(h_{11} + h_{22}).$$

Now $\ell\colon M^2 \to \mathscr{F}_y$ is clearly a section of the bundle $\mathscr{F}_X^{(1)} \to M^2$ (though not necessarily of $\mathscr{F}_X^{(\gamma)}$). We may compute Ω_X by using the identity $\Omega_X = \ell^* \circ p^*(\Omega_X)$,

$$\begin{aligned}\Omega_X &= \left(\tfrac{1}{4}(h_{11} - h_{22})^2 + h_{12}^2\right)\eta_0^1 \wedge \eta_0^2 \\ &= (H^2 - K)\eta_0^1 \wedge \eta_0^2 = (H^2 - K)\,dA.\end{aligned}$$

Thus Ω_X vanishes only along the umbilic locus of $X_0\colon M^2 \to \mathbf{E}^3$. This shows that, even though the Euclidean second fundamental form is not a conformal invariant, the notion of umbilic is a conformal invariant. This justifies our definition of the umbilic locus of X as the zero set of Ω_X. Another consequence of this calculation is that the Euclidean invariant $(H^2 - K)\,dA$ is actually a conformal invariant. This fact was noted in connection with Willmore's problem by White [11].

In order to get a section of $\mathscr{F}_X^{(\gamma)} \to M^2$, it suffices to take $\tilde{\ell} = (\tilde{e}_a)$, where

$$\begin{aligned}&\tilde{e}_0 = e_0, \qquad \tilde{e}_1 = e_1, \qquad \tilde{e}_2 = e_2, \\ &\tilde{e}_3 = e_3 + He_0, \\ &\tilde{e}_4 = y + He_3 + \tfrac{1}{2}H^2e_0 = e_4 + He_3 + \tfrac{1}{2}H^2e_0,\end{aligned}$$

for, if we now compute $\tilde{\eta}_i^3 = \tilde{h}_{ij}\tilde{\eta}_0^i = \tilde{h}_{ij}\eta_0^i$, we get

$$\begin{aligned}d\tilde{e}_3 &= de_3 + dHe_0 + Hde_0 \\ &= e_0dH - e_1\left(\eta_1^3 - H\eta_0^1\right) - e_2\left(\eta_2^3 - H\eta_0^2\right),\end{aligned}$$

so

$$\tilde{\eta}_3^0 = dH = h_1\eta_0^1 + h_2\eta_0^2,$$
$$\tilde{\eta}_1^3 = \eta_1^3 - H\eta_0^1 = \tfrac{1}{2}(h_{11} - h_{22})\eta_0^1 + h_{12}\eta_0^2,$$
$$\tilde{\eta}_2^3 = \eta_2^3 - H\eta_0^2 = h_{12}\eta_0^2 + \tfrac{1}{2}(h_{22} - h_{11})\eta_0^1.$$

Thus $\tilde{h}_{11} + \tilde{h}_{22} = 0$.

In particular, it follows that $\gamma_X = e_3 + He_0$ is the conformal Gauss map of the immersion $X = (X_0)$. This has the following geometric meaning: If $v \in Q$, then $v^\perp \subseteq \mathbf{L}^5$ is a 4-plane on which the inner product $\langle\ ,\ \rangle$ restricts to have type (3,1). In particular, $v^\perp \cap \mathcal{L}^+$ is a cone over a round $S^2 \subseteq S^3$. This S^2 has a natural orientation given by the condition that, if $\ell \in \mathcal{F}$ has $e_3(\ell) = v$, then $e_1, e_2 \bmod e_0$ form an oriented basis of $T_{(e_0)}S^2$. Conversely, every round, oriented $S^2 \subseteq S^3$ arises in this way from a unique $v \in Q$. Thus, the points of Q form the space of round, oriented S^2's in S^3. Given an immersion X: $M^2 \to S^3$ of an oriented M^2, suppose we fix $p_0 \in M$. Then there exists a 1-parameter family of round, oriented S^2's in S^3 where are oriented tangent to $X(M^2)$ at $X(p_0)$. If we think in terms of the Euclidean model $\mathbf{E}^3 = S^3 - \{(y)\}$, then these spheres are parametrized by their mean curvature ($H = 0$ corresponds to a sphere through (y), i.e. a plane in $\mathbf{E}^3$). The sphere $\gamma_X(p_0) \in Q$ is the one with the "same mean curvature" as the surface $X(M^2)$ at $X(p_0)$. Alternatively, in terms of the Euclidean model, the sphere $\gamma_X(p_0)$ is the one such that any conformal transformation which renders it into a plane transforms the surface $X(M^2)$ so that it has mean curvature zero at $X(p_0)$.

The conformal Gauss map has other interesting properties:

Proposition 2. *Let $Q \subseteq \mathbf{L}^5$ be given the induced pseudo-Riemannian structure of type* (3, 1). *Let X: $M^2 \to S^3$ be a smooth immersion of an oriented surface and endow M^2 with the induced conformal structure. Then γ_X: $M^2 \to Q$ is weakly conformal, it is an immersion away from the umbilic locus of X, and Ω_X is the induced area form of γ_X: $M^2 \to Q$.*

Proof. On $\mathcal{F}_X^{(\gamma)}$ we have $\omega_3^4 = 0$ so we compute

$$\begin{aligned}\langle de_3, de_3\rangle &= \left\langle e_0\omega_3^0 + e_1\omega_3^1 + e_2\omega_3^2, e_0\omega_3^0 + e_1\omega_3^1 + e_2\omega_3^2\right\rangle \\ &= \left(\omega_3^1\right)^2 + \left(\omega_3^2\right)^2 \\ &= \left(h_{11}\omega_0^1 + h_{12}\omega_0^2\right)^2 + \left(h_{12}\omega_0^1 + h_{22}\omega_0^2\right)^2 \\ &= \left(h_{11}^2 + h_{12}^2\right)\left(\omega_0^1\right)^2 + 2h_{12}(h_{11} + h_{22})\omega_0^1 \circ \omega_0^2 + \left(h_{12}^2 + h_{22}^2\right)\left(\omega_0^2\right)^2 \\ &= \left(h_{11}^2 + h_{12}^2\right)\left(\left(\omega_0^1\right)^2 + \left(\omega_0^2\right)^2\right)\end{aligned}$$

because $h_{11} = -h_{22}$ on $\mathscr{F}_X^{(\gamma)}$. Because $\gamma_X = e_3$ on $\mathscr{F}_X^{(\gamma)}$ and because the pseudo-metric on Q is the restriction of the pseudo-metric on $\mathbf{L}^5$, we see that $\langle d\gamma_X, d\gamma_X\rangle$ is proportional to $(\omega_0^1)^2 + (\omega_0^2)^2$, which defines the induced conformal structure on M^2. This proves that γ_X is weakly conformal. The induced area form of $\langle d\gamma_X, d\gamma_X\rangle$ is obviously

$$\left(h_{11}^2 + h_{12}^2\right)\omega_0^1 \wedge \omega_0^2 = \left(\tfrac{1}{4}(h_{11} - h_{22})^2 + h_{12}^2\right)\omega_0^1 \wedge \omega_0^2 = \Omega_X.$$

Finally, since $\Omega_X = -\omega_1^3 \wedge \omega_2^3$ on $\mathscr{F}_X^{(\gamma)}$, it follows that $\gamma_X\colon M \to Q$ is an immersion away from $\mathscr{U}_X$.

Remark. A word of caution is in order. Because the pseudo-metric on Q is not positive definite, weakly conformal is not as strong a condition on γ_X as it would be if the pseudo-metric were instead a positive definite metric.

In order to interpret the conformal transform geometrically, let us assume that our immersion $X_0\colon M^2 \to E_y$ is umbilic free and that the framing $\mathscr{E}$: $M^2 \to \mathscr{F}_y$ is *principal* (i.e. $h_{12} = 0$) with $h_{11} > h_{22}$. Then setting $R = \sqrt{\tfrac{1}{2}(h_{11} - h_{22})} > 0$, we may adapt $\tilde{\mathscr{E}}\colon M^2 \to \mathscr{F}_X^{(\gamma)}$ further to $\bar{\mathscr{E}}\colon M^2 \to \mathscr{F}_X^{(2)}$ by $\bar{\mathscr{E}} = (\bar{e}_a)$, where

$$\begin{aligned} &\bar{e}_0 = Re_0, \qquad \bar{e}_1 = e_1, \qquad \bar{e}_2 = e_2,\\ &\bar{e}_3 = (e_3 + He_0),\\ &\bar{e}_4 = R^{-1}\left(e_4 + He_3 + \tfrac{1}{2}H^2 e_0\right). \end{aligned}$$

This allows us to compute

$$d\bar{e}_3 = \bar{e}_0(R^{-1}dH) - \bar{e}_1\bar{\eta}_0^1 + \bar{e}_2\bar{\eta}_0^2.$$

So

$$\bar{\omega}_3^0 = R^{-1}dH = R^{-1}H_1\bar{\eta}_0^1 + R^{-1}H_2\bar{\eta}_0^2 = \bar{h}_1\bar{\eta}_0^1 + \bar{h}_2\bar{\eta}_0^2.$$

Thus, we may adapt $\bar{\mathscr{E}}\colon M^2 \to \mathscr{F}_X^{(2)}$ to $\mathscr{E}^*\colon M^2 \to \mathscr{F}_X^{(3)}$ by $\mathscr{E}^* = (e_a^*)$, where

$$\begin{aligned} &e_0^* = Re_0, \qquad e_1^* = e_1 - H_1 e_0,\\ &e_2^* = e_2 + H_2 e_0, \qquad e_3^* = e_3 + He_0,\\ &e_4^* = R^{-1}\left(e_4 + He_3 - H_1 e_1 + H_2 e_2 + \tfrac{1}{2}\left(H^2 + H_1^2 + H_2^2\right)e_0\right). \end{aligned}$$

As a consequence, we have $\omega_3^{0*} = 0$.

The geometric meaning of $\hat{X} = (e_4^*)$ is now clear: If $p \in M$ is nonumbilic for $X\colon M^2 \to S^3$, then $\hat{X}(p)$ is the unique point in S^3 such that a stereographic projection from $\hat{X}(p)$, $\rho\colon S^3 - \{X(p)\} \to \mathbf{E}^3$, causes the mean curvature of the immersion $\rho \circ X\colon M^2 \to \mathbf{E}^3$ to vanish to *second* order at $p \in M^2$.

2. The variational equations for the Willmore functional

If X: $M^2 \to S^3$ is a smooth immersion of an oriented surface, we have seen that we may construct a canonical 2-form Ω_X ($\geqslant 0$) on M^2 from the second order jet of X. If $K \subseteq M^2$ is a compact domain in M, we can define the functional

$$\mathscr{W}_K(X) = \int_K \Omega_X$$

on the space of smooth immersions of M^2 into S^3. We say that X is a *Willmore immersion* if for any compact $K \subseteq M$ and any smooth variation X_t: $M \to S^3$ with support in K, we have

$$\frac{d}{dt}(\mathscr{W}_K(X_t))\bigg|_{t=0} = 0$$

(of course, $X_0 = X$). The purpose of this section is to calculate the Euler-Lagrange equation for this variational problem in a conformally invariant way. We do this by the method of moving frames.

Let X_t: $M^2 \to S^3$ be a smooth 1-parameter family of immersions with support in a compact set $K \subseteq M$ for $|t| < \varepsilon$ for some $\varepsilon > 0$. We may assume that the variation is normal since the support is compact. It follows that we may construct a G_γ-bundle $\mathscr{F}_{X_t}^{(\gamma)} \subseteq M \times (-\varepsilon, \varepsilon) \times \mathscr{F}$ with the property that

$$\mathscr{F}_{X_{t_0}}^{(\gamma)} = \left\{(p, t_0, \ell) \in \mathscr{F}_{X_t}^{(\gamma)}\right\}$$

for all $t_0 \in (-\varepsilon, \varepsilon)$. (We cannot construct $\mathscr{F}^{(2)}$, etc., without making an umbilic-free assumption).

The important defining properties of $\mathscr{F}_{X_t}^{(\gamma)}$ are that

(i) $\omega_0^3 = \lambda\, dt$ for some smooth function λ on $\mathscr{F}_{X_t}^{(\gamma)}$.

(ii) $\omega_0^1 \wedge \omega_0^2$ is semibasic for the projection $\mathscr{F}_{X_t}^{(\gamma)} \to M$ and is positive. (This uses the normality of the variation.)

(iii) $\omega_i^3 = h_{ij}\omega_0^j + \lambda_i\, dt$, $d\lambda = -\lambda\omega_0^0 + \lambda_i\omega_0^i + \lambda'\, dt$, where $h_{ij} = h_{ji}$, λ_i, λ' are smooth functions on $\mathscr{F}_{X_t}^{(\gamma)}$.

We also have $h_{11} + h_{22} \equiv 0$. (These equations come from differentiating the equation $\omega_0^3 = \lambda\, dt$ and using the structure equations. The normalization $h_{11} + h_{22} \equiv 0$ is the equation defining $\mathscr{F}_{X_t}^{(\gamma)} \subseteq \mathscr{F}_{X_t}^{(1)}$.) Now λ is well defined up to a multiple on $M \times (-\varepsilon, \varepsilon)$ and its support lies above $K \times (-\varepsilon, \varepsilon)$ in $\mathscr{F}_{X_t}^{(\gamma)}$ since λ vanishes when the variation vectorfield vanishes. Also, the support of the functions λ_i lies above $K \times (-\varepsilon, \varepsilon)$ because of the equation $d\lambda + \lambda\omega_0^0 = \lambda_i\omega_0^i + \lambda'\, dt$. Note that we have

$$\Omega_{X_{t_0}} = \left(-\omega_1^3 \wedge \omega_2^3\right)\big|_{t=t_0}$$

for all t_0. It follows that if we set

$$\Phi = -\omega_1^3 \wedge \omega_2^3 + dt \wedge \left(\partial/\partial t \lrcorner \left(\omega_1^3 \wedge \omega_2^3\right)\right),$$

then $\partial/\partial t \lrcorner \Phi = 0$ and we have, for all t_0,

$$\Omega_{X_{t_0}} = \Phi|_{t=t_0}.$$

This implies that Φ is well defined on $M \times (-\varepsilon, \varepsilon)$ and is semibasic for the projection to M. We may then set

$$f(t_0) = \mathscr{W}_K\left(X_{t_0}\right) = \int_K \Phi \bigg|_{t=t_0}.$$

We compute the variation

$$f'(0) = \int_K \left(\mathscr{L}_{\partial/\partial t}(\Phi)\right)\bigg|_{t=0} = \int (\partial/\partial t \lrcorner d\Phi)\bigg|_{t=0}$$

(since $\partial/\partial t \lrcorner \Phi \equiv 0$). This expands to

$$f'(0) = \int_K \left(-\partial/\partial t \lrcorner \left(d\omega_1^3 \wedge \omega_2^3 \wedge -\omega_1^3 \wedge d\omega_2^3\right) - d\left(\lambda_1 \omega_2^3 \wedge \lambda_2 \omega_1^3\right)\right)\bigg|_{t=0}.$$

By the structure equations,

$$-d\left(\omega_1^3 \wedge \omega_2^3\right) = \omega_0^3 \wedge \left(\omega_1^0 \wedge \omega_2^3 + \omega_1^3 \wedge \omega_2^0\right) + \omega_3^0 \wedge \left(\omega_0^1 \wedge \omega_2^3 + \omega_1^3 \wedge \omega_0^2\right).$$

If we denote restriction to $t = 0$ by an overbar, we find

$$-\left(\partial/\partial t \lrcorner d\left(\omega_1^3 \wedge \omega_2^3\right)\right)\big|_{t=0} = \bar{\lambda} \wedge \left(\bar{\omega}_1^0 \wedge \bar{\omega}_2^3 + \bar{\omega}_1^3 \wedge \bar{\omega}_2^0\right) + \bar{\omega}_3^0 \wedge \left(\bar{\lambda}_2 \bar{\omega}_0^1 - \bar{\lambda}_1 \bar{\omega}_0^2\right).$$

This may be rewritten as follows: We know that $\bar{\omega}_3^0 = \bar{h}_i \bar{\omega}_0^i$ and, if we differentiate this, use the structure equations, and apply Cartan's Lemma, we get

$$d\bar{h}_i = 2\bar{h}_i \bar{\omega}_0^0 + \bar{h}_j \bar{\omega}_i^j + \bar{h}_{ij} \bar{\omega}_j^0 + \bar{p}_{ij} \bar{\omega}_0^j$$

for some smooth functions $\bar{p}_{ij} = \bar{p}_{ji}$ on $\mathscr{F}_{X_0}^{(\gamma)}$. It is then an elementary matter to compute that

$$-\left(\partial/\partial t \lrcorner d\left(\omega_1^3 \wedge \omega_2^3\right)\right)\big|_{t=0} = \bar{\lambda}\left(\bar{p}_{11} + \bar{p}_{22}\right)\bar{\omega}_0^1 \wedge \bar{\omega}_0^2 - d\left(\bar{\lambda}\left(\bar{h}_1 \bar{\omega}_0^2 - \bar{h}_2 \bar{\omega}_0^1\right)\right).$$

Thus, our formula becomes

$$f'(0) = \int_K \lambda\left(\bar{p}_{11} + \bar{p}_{22}\right)\bar{\omega}_0^1 \wedge \bar{\omega}_0^2 - d\chi,$$

where

$$\chi = \bar{\lambda}_1 \bar{\omega}_2^3 - \bar{\lambda}_2 \bar{\omega}_1^3 + \bar{\lambda}\left(\bar{h}_1 \bar{\omega}_0^2 - \bar{h}_2 \bar{\omega}_0^1\right).$$

It is now a routine matter to check that χ is a well-defined, smooth 1-form on M and by our previous discussion of λ and λ_i, χ is supported in K. Thus, by Stoke's Theorem, we get the variational formula

$$f'(0) = \int_K \bar{\lambda}(\bar{p}_{11} + \bar{p}_{22})\bar{\omega}_0^1 \wedge \bar{\omega}_0^2.$$

This motivates the following definition. Let X: $M^2 \to S^3$ be a smooth immersion and let $\mathscr{F}_X^{(\gamma)} \to M^2$ be its γ-order frame bundle. On $\mathscr{F}_X^{(\gamma)}$, we have $\omega_3^0 = h_i \omega_0^i$ and the structure equations imply that there exist smooth functions $p_{ij} = p_{ji}$ on $\mathscr{F}_X^{(\gamma)}$ so that

$$dh_i = 2h_i\omega_0^0 + h_j\omega_i^j + h_{ij}\omega_j^0 + p_{ij}\omega_0^j.$$

Our discussion above implies that the two-form $\delta\Omega_X = (p_{11} + p_{22})\omega_0^1 \wedge \omega_0^2$ is well defined up to a positive factor on M (in any case, this is elementary to check). We call $\delta\Omega_X$ the *first variation* of the Willmore integrand, Ω_X. Note that $\delta\Omega_X$ depends on the *fourth* order jet of X, while Ω_X depends on the second order jet. We record this as

Theorem A. *Let M^2 be an oriented surface. Then X: $M^2 \to S^3$ is a Willmore immersion if and only if $\delta\Omega_X \equiv 0$.*

Proof. If λ is any smooth function on M with compact support in $K \subseteq M$, then there exists a variation X_t with support in K so that λe_3 is the variation vectorfield at $t = 0$. Then we have shown that

$$\frac{d}{dt}(\mathscr{W}(X_t))_{|t=0} = \int_K \lambda\,\delta\Omega_X.$$

Since λ is arbitrary with compact support, it follows that the left-hand side vanishes for all such variations iff $\delta\Omega_X \equiv 0$. q.e.d.

A few remarks are in order concerning the geometric meaning of the condition $\delta\Omega_X = 0$. Since these will not be needed in the sequel, we leave the proofs as exercises in the use of the structure equations.

The first remark is that, if we pursue the relationship with Euclidean surface theory begun in §1, we can easily show that the condition $\delta\Omega_X = 0$ is equivalent to

$$\Delta H + 2(H^2 - K)H = 0$$

when we regard X: $M^2 \to S^3$ as arising from an Euclidean immersion X_0: $M^2 \to \mathbf{E}^3$. In this form, the Euler-Lagrange equation for the integrand $\int(H^2 - K)\,dA$ has been known for some time [12].

The second remark concerns the relationship of this problem with the geometry of the conformal Gauss map γ_X: $M^2 \to Q^4$. For definiteness, let us assume X: $M^2 \to Q^4$ is free of umbilics. Then γ_X: $M^2 \to Q^4$ is a space-like immersion and Ω_X is the area form induced from the pseudo-metric on Q^4. From this it is obvious that if γ_X is a minimal immersion, then X must be Willmore. Interestingly enough, the converse is also true. We see this as follows: If we compute the signature of the inner product on the normal bundle to the immersion γ_X, we see that it is of type $(1,1)$. In fact, because of our umbilic-free assumption, the normal plane to γ_X at $p \in M$ is the 2-plane spanned by $X(p)$ and $\hat{X}(p)$ (the conformal transform). By the structure equations, the mean curvature of the immersion γ_X in the direction X is zero already. The condition that the mean curvature in the direction $\hat{X}$ be zero is exactly that $\delta\Omega_X \equiv 0$. This establishes our claim. In some sense, the theory of minimal space-like surfaces in Q^4 and the Willmore surfaces in S^3 are the "same" theory, at least under suitable nondegeneracy hypotheses. This situation has been encountered before in the relationship of surface theory in S^4 (instead of Q^4) with the theory of complex curves in $Q_3(\mathbf{C})$, the complex 3-quadric (instead of real surfaces in S^3). This should be no surprise since SO(5) and SO(4, 1) are merely different real forms of the same complex Lie group. It appears that all of this must fit into a sort of generalized twistor program, but we will not pursue this any further.

Our third and final remark is related to the second. If γ_X: $M^2 \to Q^4$ is minimal and $U \subseteq \mathcal{N}_X$ $(\subseteq M)$ is the open set, where $\hat{X}$: $\mathcal{N}_X \to S^3$ is an immersion, then $\gamma_{\hat{X}}$: $U \to Q^4$ is given by the formula $\gamma_{\hat{X}} = -\gamma_X$. It follows that $\hat{X}$: $U \to S^3$ is a Willmore immersion and that $\hat{\hat{X}} = X$. In this case, we say that $\hat{X}$ is the *Willmore dual* of X. In the next section, we are going to show that $\hat{X}$ extends to be a smooth map $\hat{X}$: $M^2 \to S^3$ and is, in fact, a *conformal branched immersion*, where we use the conformal structure on M^2 induced by X: $M^2 \to S^3$.

3. The conformal structure and some complex geometry

Throughout this section X: $M^2 \to S^3$ will be a Willmore immersion of an oriented surface M^2 and $\mathcal{F}_X^{(\gamma)} \to M^2$ will denote the G_γ-bundle of γ-order frames. The quadratic form $(\omega_0^1)^2 + (\omega_0^2)^2$ is well defined up to a positive multiple on M^2 and hence induces a well-defined conformal structure on M^2. Moreover, the orientation of M^2 is represented by the positivity of the 2-form $\omega_0^1 \wedge \omega_0^2$ (again only defined up to positive multiples on M^2). It follows from the existence of isothermal coordinates (see [3]) that M^2 possesses a unique complex structure compatible with the given conformal structure and orientation. The defining property of this complex structure is that a complex valued 1-form η on M^2 is of type $(1,0)$ iff $p^*(\eta) = a(\omega_0^1 + i\omega_0^2)$ for some complex function a on $\mathscr{F}_X^{(Y)}$ (where p is the projection $p\colon \mathscr{F}_X^{(Y)} \to M^2$). From now on, we regard M^2 as a complex curve with this given complex structure.

Retaining the notation from §§1, 2, we introduce the complex notation for forms on $\mathscr{F}_X^{(Y)}$:

$$(3.1) \qquad \omega = \omega_0^1 + i\omega_0^2, \qquad \alpha = \omega_1^0 + i\omega_2^0, \qquad \phi = \omega_0^0, \qquad \rho = \omega_1^2,$$

$$(3.2) \qquad z = h_{11} - ih_{12}, \qquad h = \tfrac{1}{2}(h_1 - ih_2).$$

Then the formulae in §1 become

$$(3.3) \qquad \omega_1^3 - i\omega_2^3 = z\omega,$$

$$(3.4) \qquad \omega_3^0 = h\omega + \bar{h}\bar{\omega},$$

$$(3.5) \qquad dz = (\phi + 2i\rho)z + \zeta\omega + h\bar{\omega},$$

where ζ is a linear combination of the h_{ijk} (the coefficients are unimportant for what follows). Differentiating (3.4) and using the structure equations give

$$(3.6) \qquad dh = (2\phi + i\rho)h + \tfrac{1}{2}z\alpha + q\omega,$$

where $q = (p_{11} - ip_{12})$, as the $p_{ij} = p_{ji}$ were defined in §2. Note that (3.6) uses the fact that X is a Willmore immersion, i.e., $p_{11} + p_{22} = 0$. Otherwise, we would have had to add the term $\frac{1}{2}(p_{11} + p_{22})\bar{\omega}$ to the right-hand side of (3.6). For convenience, we list the following consequences of the structure equations in this notation:

$$(3.7) \qquad d\omega = -(\phi + i\rho) \wedge \omega,$$

$$(3.8) \qquad d\alpha = (\phi - i\rho) \wedge \alpha - h\bar{z}\omega \wedge \bar{\omega},$$

$$(3.9) \qquad d\phi = \tfrac{1}{2}(\bar{\alpha} \wedge \omega + \alpha \wedge \bar{\omega}),$$

$$(3.10) \qquad d\rho = (i/2)z\bar{z}\omega \wedge \bar{\omega} + (i/2)(\omega \wedge \bar{\alpha} + \alpha \wedge \bar{\omega}).$$

Theorem B. *There exists a holomorphic quartic form $\mathscr{Q}_X$ on M defined by the condition*

$$p^*(\mathscr{Q}_X) = (zq - h\zeta)(\omega)^4.$$

(Note that we use the symmetric product $(\omega)^4 = \omega \circ \omega \circ \omega \circ \omega$. Thus, we are asserting that $\mathscr{Q}_X$ as defined above is a holomorphic section of the fourth power of the canonical bundle over M.)

Proof. If we differentiate equations (3.5) and (3.6) using the identities (3.5)–(3.10) and applying Cartan's Lemma, we see that there must exist smooth functions r, s on $\mathscr{F}_X^{(Y)}$ satisfying

$$(3.11) \qquad d\zeta = (2\phi + 3i\rho)\zeta - \tfrac{3}{2}z\bar{\alpha} - z^2\bar{z}\bar{\omega} + r\omega + q\bar{\omega},$$

$$(3.12) \qquad dq = (3\phi + 2i\rho)q + \tfrac{1}{2}\zeta\alpha - \tfrac{3}{2}h\bar{\alpha} - hz\bar{z}\bar{\omega} + s\omega.$$

But then, equations (3.5), (3.6), (3.11) and (3.12) combine to give

$$d(zq - h\zeta) = 4(\phi + i\rho)(zq - h\zeta) + (zs - hr)\omega. \tag{3.13}$$

Now, if f and g were *any* smooth functions on $\mathscr{F}_X^{(Y)}$ and $n \geq 0$ were any integer satisfying the equation

$$df = h(\phi + i\rho)f + g\omega, \tag{3.14}$$

then the quantity $f(\omega)^n$ would be semibasic and locally constant on the fibers of $\mathscr{F}_X^{(Y)} \to M$ (by (3.7) and (3.14)). Since G_Y is connected, this would be enough to ensure that there existed an $\mathfrak{F}$ on M so that $p^*(\mathfrak{F}) = f(\omega)^n$. Moreover, we claim that this $\mathfrak{F}$ would necessarily be a *holomorphic* section of the nth power of the canonical bundle of M. It suffices to check this locally, so let $m \in M$ be fixed and let ξ: $U \to \mathbf{C}$ be a holomorphic coordinate chart on a neighborhood U of m. Then $d\xi$ is of type (1, 0) so it follows that there is a section over U, say σ: $U \to \mathscr{F}_X^{(Y)}$, satisfying $\sigma^*(\omega) = d\xi$. Then

$$\begin{aligned} 0 = d^2\omega = \sigma^*(d\omega) &= -\sigma^*(\phi + i\rho) \wedge \sigma^*(\omega) \\ &= -\sigma^*(\phi + i\rho) \wedge d\xi, \end{aligned}$$

so $\sigma^*(\phi + i\rho) = a\, d\xi$ for some $a \in C^\infty(U)$ (complex valued, of course). Because σ is a section,

$$\mathfrak{F}_{|U} = \sigma^*(f(\omega^n)) = (f \circ \sigma)(d\xi)^n.$$

Finally, (3.14) implies $d(f \circ \sigma) = (g + naf)\, d\xi$, so $\partial(f \circ \sigma)/\partial\bar{\xi} = 0$.

In other words, $f \circ \sigma$ is a holomorphic function on U, so $\mathfrak{F}_{|U} = (f \circ \sigma)(d\xi)^n$ is a holomorphic section of the nth power of the canonical bundle restricted to U. Since $m \in M$ was arbitrary, it follows that $\mathfrak{F}$ is holomorphic.

Now all that we have said applies in the case where $n = 4$, $f = zq - h\zeta$, $g = zs - hr$ and $\mathfrak{F} = \mathscr{Q}_X$.

Theorem C. *Let M^2 be connected and let X: $M^2 \to S^3$ be a Willmore immersion. Either $\mathscr{U}_X \equiv M^2$ or else $\mathscr{U}_X$ is a closed subset of M^2 with no interior. In this latter case, the conformal transform extends uniquely and smoothly to a map $\hat{X}$: $M^2 \to S^3$. If $\mathscr{Q}_X \equiv 0$, then $\hat{X}$ is a constant map. If $\mathscr{Q}_X \not\equiv 0$, then $\hat{X}$: $M^2 \to S^3$ is a conformal branched immersion where the branching order of $\hat{X}$ at $m \in M^2$ is less than or equal to the vanishing order of $\mathscr{Q}_X$ at m.*

Proof. Consider the map Y: $\mathscr{F}_X^{(Y)} \to \mathbf{L}^5$ given by

$$Y = 2h\bar{h}e_0 - \bar{z}he - z\bar{h}\bar{e} + z\bar{z}e_4, \tag{3.15}$$

where $e = e_1 - ie_2$. We compute that

$$\langle Y, Y \rangle = 0 \tag{3.16}$$

and the structure equations show that

$$dY = 3Y\phi + Z\omega + \bar{Z}\bar{\omega}, \tag{3.17}$$

where

$$Z = 2\bar{h}qe_0 - \bar{z}qe - \bar{h}\zeta\bar{e} + \bar{z}\zeta e_4. \tag{3.18}$$

We have the identities

$$\langle Y, Z\rangle = \langle Z, Z\rangle = 0, \qquad \langle Z, \bar{Z}\rangle = 2(zq - h\zeta)\overline{(zq - h\zeta)}. \tag{3.19}$$

In particular,

$$\langle dY, dY\rangle = 4|zq - h\zeta|^2 \omega \circ \bar{\omega}. \tag{3.20}$$

Note that $\mathscr{F}_X^{(3)} \subseteq \mathscr{F}_X^{(\gamma)}$ is defined by the equation $h = 0$, $z = 1$, so $Y = e_4$ on $\mathscr{F}_X^{(3)}$ (if $\mathscr{F}_X^{(3)} \neq \varnothing$). Now equation (3.17) shows that, on any fiber of $\mathscr{F}_X^{(\gamma)} \to M$, Y only varies by a positive multiple, so, in particular (Y): $\mathscr{N}_X \to S^3$ is well defined (since Y: $\mathscr{F}_X^{(\gamma)}(\mathscr{N}_X) \to \mathscr{L}^+$) and is equal to the conformal transform $\hat{X}$: $\mathscr{N}_X \to S^3$.

We are going to show that either $Y \equiv 0$ (so that $\mathscr{U}_X = M$) or that there exists a smooth Y_0: $\mathscr{F}_X^{(\gamma)} \to \mathscr{L}^+$ and a smooth nonnegative function with isolated zeros, λ, on M so that $Y = \lambda Y_0$. It will follow that (Y_0): $M^2 \to S^3$ is a smooth extension of (Y): $\mathscr{N}_X \to S^3$ and hence of $\hat{X}$. Uniqueness and the rest of the properties claimed for this extension will follow from our construction of Y_0.

First we must prove a few facts.

Fact 1. If $U \subseteq M$ is an open set with a holomorphic coordinate chart ξ: $U \to \mathscr{D} \subseteq \mathbf{C}$, then there exists a unique section σ: $U \to \mathscr{F}_X^{(\gamma)}$ satisfying $\sigma^*(\omega) = d\xi$ and $\sigma^*(\phi + i\rho) = 0$.

Proof. Because $d\xi$ is of type $(1,0)$ and nonvanishing on U, there exists a smooth (complex-valued) function a on $\mathscr{F}_X^{(\gamma)}(U)$ so that $a \neq 0$ and

$$p^*(d\xi) = a\omega. \tag{3.21}$$

Taking the exterior derivative of (3.21) and using (3.7), we see that there exists a smooth function b on $\mathscr{F}_X^{(\gamma)}(U)$ so that

$$da = (\phi + i\rho)a + b\omega. \tag{3.22}$$

Taking the exterior derivative of (3.22) and using (3.7), (3.9) and (3.10) we see that there exists a smooth function c on $\mathscr{F}_X^{(\gamma)}(U)$ so that

$$db = 2(\phi + i\rho)b - a\bar{\alpha} + c\omega - \tfrac{1}{2}az\bar{z}\bar{\omega}. \tag{3.23}$$

It follows that there exists a unique section σ: $U \to \mathscr{F}_X^{(\gamma)}(U)$ satisfying $\sigma^*(a) = 1$ and $\sigma^*(b) = 0$. Applying this σ to (3.21) and (3.22) gives the desired result.

Fact 2. Let $m \in M$ be fixed and let U be a connected open neighborhood of m on which there exists a coordinate chart ξ: $U \to \mathscr{D} \subseteq \mathbf{C}$ with $\xi(m) = 0$.

Let $\sigma\colon U \to \mathscr{F}_X^{(\gamma)}(U)$ be the section defined above by Fact 1. Then either the functions $z \circ \sigma$ and $h \circ \sigma$ vanish identically on U or there exists a nonnegative integer k and smooth functions z_1, h_1 on U with $(z_1(m), h_1(m)) \neq (0,0)$ so that

$$z \circ \sigma = \xi^k z_1, \qquad h \circ \sigma = \xi^k h_1. \tag{3.24}$$

Proof. Applying σ^* to (3.23) we get

$$\sigma^*(\alpha) = \overline{(c \circ \sigma)}\, \overline{d\xi} - \tfrac{1}{2}(z \circ \sigma)\overline{(z \circ \sigma)}\, d\xi. \tag{3.25}$$

If we apply σ^* to (3.5) and (3.6) we get

$$\begin{aligned} d(z \circ \sigma) &= (\zeta \circ \sigma)\, d\xi + (h \circ \sigma)\, d\bar{\xi}, \\ d(h \circ \sigma) &= \left(q \circ \sigma - \tfrac{1}{4}(z \circ \sigma)^2\overline{(z \circ \sigma)}\right) d\xi + \tfrac{1}{2}\overline{(c \circ \sigma)}(z \circ \sigma)\, d\bar{\xi}. \end{aligned} \tag{3.26}$$

In particular, we have

$$\frac{\partial}{\partial \bar{\xi}}\begin{pmatrix} z \circ \sigma \\ h \circ \sigma \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \tfrac{1}{2}\overline{(c \circ \sigma)} & 0 \end{pmatrix}\begin{pmatrix} z \circ \sigma \\ h \circ \sigma \end{pmatrix}. \tag{3.27}$$

It is now an elementary consequence of the Newlander-Nirenberg theorem (see [10]) that (3.27) implies the conclusion of Fact 2.

Remark. If $z \circ \sigma$ and $h \circ \sigma$ vanish identically on U we set $k(m) = \infty$, otherwise, we let $k(m) > 0$ be the integer defined in Fact 2. It is an elementary matter to check that $k(m)$ is actually well defined, i.e. depends only on m.

The proof of Fact 2 actually shows that the sets

$$U_\infty = \{m \in M \,|\, k(m) = \infty\}, \qquad U_f = \{m \in M \,|\, k(m) < \infty\}$$

are open, disjoint and cover M. It follows that one must be empty (since M is connected). If $U_f = \varnothing$, then $z \equiv 0$ on $\mathscr{F}_X^{(\gamma)}$ so $\mathscr{U}_X \equiv M$. We now set this case aside and we assume $U_\infty = \varnothing$.

Again, the proof of Fact 2 shows that the set $U_+ = \{m \in M \,|\, k(m) > 0\}$ is a discrete set and hence that

$$D = \sum_{m \in M} k(m) \cdot m$$

is a divisor in M. We are now going to show that $\mathscr{U}_X$ has no interior. Indeed, if $m \in M$, then selecting U and σ as above, we may write $z \circ \sigma = \xi^k z_1$. If $z_1(m) \neq 0$, then the only possible zero of $z \circ \sigma$ on a sufficiently small neighborhood of m is m itself since ξ vanishes only at m. If $z_1(m) = 0$, then $h_1(m) \neq 0$ and (3.27) implies $\partial z_1/\partial\bar{\xi} = h_1$, so $dz_1 \neq 0$ at m. It follows that $z_1^{-1}(0)$ has no interior on a neighborhood of m (sufficiently small). Thus the locus $z \circ \sigma = 0$ on a neighborhood $U' \subseteq U$ has no interior, so we are done. The fact that $\mathscr{U}_X$ is closed is obvious.

Now for each $m \in U_+$, choose an open disk $\Delta_m \subseteq M$ with $m \in \Delta_m$ in such a way that $\bar{\Delta}_m \cap \bar{\Delta}_n = \varnothing$ for $m \neq n$. Let $\xi_m\colon \Delta_m \to \Delta$ be a holomorphic coordinate, where $\xi_m(m) = 0$ and $\Delta = \{a \in \mathbf{C} \mid |a| < 1\}$ is the unit disk in $\mathbf{C}$. By a partition of unity argument, we may construct a smooth real-valued function λ on M with the property that on each disk λ_m we have $\lambda|_{\Delta_m} = (\xi_m\bar{\xi}_m)^{k(m)}$ and $\lambda(m) > 0$ if $m \notin U_+$.

Fact 3. The map $(\lambda \circ p)^{-1}Y$ has a unique smooth extension, Y_0, across the fibers $p^{-1}(U_+) \subseteq \mathscr{F}_X^{(\gamma)}$.

Proof. Because U_+ is discrete in M, the compliment of $p^{-1}(U_+)$ is dense in $\mathscr{F}_X^{(\gamma)}$. Thus if there is a smooth extension it is unique. To show that Y_0 exists it clearly suffices to show that, for each $m \in U_+$, $\lambda^{-1}(Y\sigma_m)$ has a smooth extension across m, where $\sigma_m\colon \Delta_m \to \mathscr{F}_X^{(\gamma)}(\Delta_m)$ is the section in Fact 1. When we restrict to Δ_m, $\lambda = (\xi_m\bar{\xi}_m)^{k(m)}$, so

$$\lambda^{-1}(Y \circ \sigma_m) = 2h_1\bar{h}_1(e_0 \circ \sigma) - \bar{z}_1h_1\overline{(e \circ \sigma)} + z_1\bar{z}_1(e_4 \circ \sigma), \tag{3.28}$$

which is clearly a smooth map from Δ_m to $\mathscr{L}^+$. q.e.d.

From now on, we refer to this smooth extension as $Y_0\colon \mathscr{F}_X^{(\gamma)} \to \mathscr{L}^+$. It is now a routine (albeit tedious) matter to show that if we set $d(\lambda \circ p) = \mu\omega + \bar{\mu}\bar{\omega}$ and $Z_0 = (\lambda \circ p)^{-1}(Z - \mu Y_0)$, then $Z_0\colon \mathscr{F}_X^{(\gamma)} \to \mathbf{L}^5$ is also smooth and we have the relations

$$\langle Y_0, Y_0\rangle = 0, \qquad \langle Y_0, Z_0\rangle = 0, \qquad \langle Z_0, Z_0\rangle = 0,$$

$$\langle Z_0, \bar{Z}_0\rangle = 2|zq - h\zeta|^2(\lambda \circ p)^{-2}, \qquad dY_0 = 3Y_0\phi + Z_0\omega + \bar{Z}_0\bar{\omega}.$$

Fact 4. If $\mathscr{Q}_X \equiv 0$, then $(Y_0)\colon M^2 \to S^3$ is a constant map.

Proof. It clearly suffices to prove that $(Y)\colon M^2 - U_f \to S^3$ is a constant map. To do this, it is sufficient to note that

$$Y \wedge Z = -(zq - h\zeta)(\bar{h}\bar{z})(2e_0 \wedge e_4 + e \wedge \bar{e}) \equiv 0$$

if $\mathscr{Q}_X = 0$, so $Y \wedge dY \equiv 0$ on $\mathscr{F}_X^{(\gamma)}$. This is well known to imply that (Y): $M^2 \to U_+ \to S^3$ is constant. q.e.d.

Now suppose $\mathscr{Q}_X \not\equiv 0$.

Fact 5. If $\nu(m)$ is the vanishing order of $\mathscr{Q}_X$ at m, then $\nu(m) \geqslant 2k(m)$.

Proof. If $m \notin U_+$, there is nothing to prove, so assume $k(m) > 0$. Then, in the notation of the proof of Fact 2, we see that

$$\zeta \circ \sigma = \partial(z \circ \sigma)/\partial\xi = k\xi^{k-1}z_1 + \xi^k\partial z_1/\partial\xi,$$

$$\begin{aligned} q \circ \sigma &= (h \circ \sigma)/\partial\xi + \tfrac{1}{4}(z \circ \sigma)^2\overline{(z \circ \sigma)} \\ &= h\xi^{k-1}h_1 + \xi^k\partial h_1/\partial\xi + \tfrac{1}{4}\xi^{2k}\bar{\xi}^k z_1^2\bar{z}_1. \end{aligned}$$

But then, on U, we have

$$\begin{aligned}\mathcal{Q}_X &= \sigma^*\big((zq - h\zeta)(\omega)^4\big)\\ &= \xi^{2k}\big(z_1(\partial h_1/\partial\xi) - h_1(\partial z_1/\partial\xi) + \tfrac{1}{4}\bar{\xi}^k z_1^3 \bar{z}_1\big)(d\xi)^4.\end{aligned}$$

Since $\xi(m) = 0$ but $d\xi \neq 0$, the claim follows.

It follows, since

$$\langle dY_0, dY_0\rangle = 4(\lambda \circ p)^{-2}|zq - h\zeta|^2 \omega \circ \bar{\omega},$$

that (Y_0): $M^2 \to S^3$ is a branched conformal immersion with branching order $\nu(m) - 2k(m)$ at $m \in M$ (see [6]). This concludes the proof of Theorem C.

4. The spherical Willmore surfaces

In this section, we prove our main theorem concerning the Willmore immersions X: $S^2 \to S^3$. The starting point is

Theorem D. *Let X: $S^2 \to S^3$ be a Willmore immersion. Then $\mathcal{Q}_X \equiv 0$. Thus, either X is all umbilic, so that $X(S^2) \subseteq S^3$ is a round 2-sphere, or else $\hat{X}$: $S^2 \to S^3$ is a constant map.*

Proof. The form $\mathcal{Q}_X$ is a holomorphic section of κ^4, where κ is the canonical bundle of $S^2 = \mathbf{P}^1$. It is well known that $\kappa \simeq \mathcal{O}(-2)$, so $\kappa^4 \simeq \mathcal{O}(-8)$. In particular κ^4 is a negative bundle so any holomorphic section of κ^4 vanishes identically. Thus $\mathcal{Q}_X \equiv 0$. The rest follows from Theorem C.

Theorem E. *Let X: $M^2 \to S^3$ be a Willmore immersion of a compact connected surface M^2. Assume that X is not all umbilic but that $\mathcal{Q}_X \equiv 0$. Let $\hat{X} \equiv (y_0) \in S^3$, let $D = X^{-1}((y_0)) \subseteq M^2$ and let $M^* = M - D$. Then D is a nonempty finite discrete set. If ρ: $S^3 - (y_0) \to \mathbf{E}^3$ is a stereographic projection, then $\rho \circ X$: $M^* \to \mathbf{E}^3$ is a complete minimal immersion with finite total curvature. Its ends are imbedded and have zero logarithmic growth.*

Proof. The fact that D is discrete (and hence finite) follows from the fact that X is an immersion. The fact that $\rho \circ X$: $M^* \to \mathbf{E}^3$ is minimal follows from our discussion in §1, since, for each $m \in M^*$, $\hat{X}(m) = (y_0) \neq X(m)$ and we saw that the stereographic projection $S^3 - \{\hat{X}(m)\} \to \mathbf{E}^3$ caused the mean curvature of the image surface to vanish to second order at m. Thus the mean curvature H of $\rho \circ X$: $M^* \to \mathbf{E}^3$ vanishes to second order at every $m \in M^*$ and hence must vanish identically on M^*. Since there are no compact minimal surfaces in $\mathbf{E}^3$, it follows that M^* cannot be all of M, i.e., $D \neq \varnothing$.

The completeness of $\rho \circ X$: $M^* \to \mathbf{E}$ is almost obvious since it suffices to show that if $\{m_i\}$ is a sequence in M^* converging to $p_0 \in D$, then $\rho \circ X(m_i)$ diverges to ∞ in $\mathbf{E}^3$. But, if $m_i \to p_0$, then $X(m_i) \to X(p_0) = (y_0)$ so

$\rho \circ X(m_i) \to \infty$ in $\mathbf{E}^3$. Finite total curvature follows from the identity in §1, that

$$\int_{M^*} (-K)\, dA = \int_{M^*} (H^2 - K)\, dA = \int_{M^*} \Omega_X = \int_M \Omega_X < \infty,$$

where dA is the induced area form on M^* for $\rho \circ X\colon M^* \to \mathbf{E}^3$ and K is the Gauss curvature of the induced metric. The first equality above follows from $H = 0$, the second from §1, the third follows from the finiteness of D, and the finiteness at the last stage follows from the compactness of M.

Clearly the ends of M^* are in one-to-one correspondence with the points of D. In fact, if $m_0 \in D$, then there exists a disk $\Delta_0 \subseteq M$ which is an open neighborhood of m_0 and on which $X\colon \Delta_0 \to S^3$ is an embedding. But then $\rho \circ X\colon \Delta_0 - \{m_0\} \to \mathbf{E}^3$ is an imbedding. This shows that the end at m_0 is imbedded. Since $m_0 \in D$ was arbitrary, it follows that $\rho \circ X\colon M^* \to \mathbf{E}^3$ has imbedded ends.

By a theorem of Osserman [9], it follows that $\partial(\rho \circ X)$ is a meromorphic $\mathbf{C}^3$-valued 1-form (of type $(1,0)$) on M with poles exactly on D. The fact that each end is imbedded implies that the order of the pole at $m_0 \in D$ is exactly 2. Let $\Delta_0 \subseteq M$ be an open disk containing m_0 (and no other point of D) and let $\xi\colon \Delta_0 \to \{a \in \mathbf{C} \mid |a| < 1\}$ be a holomorphic coordinate with $\xi(m_0) = 0$. It follows that there exist vectors $\{v_i \in \mathbf{C}^3 | -2 \leqslant i < \infty\}$ with $v_{-2} \neq 0$ so that on Δ_0 we have a series expansion

$$\partial(\rho \circ X) = \left(\xi^{-2} v_{-2} + \xi^{-1} v_{-1} + v_0 + \cdots\right) d\xi.$$

This implies (by the reality of $\rho \circ X$) that, restricted to $\Delta_0 - \{m_0\}$,

$$\rho \circ X = \mathrm{Re}\left(-\xi^{-1} v_{-2} + \log(\xi) v_{-1} + v_0 \xi + \cdots\right) + V,$$

where V is a constant vector in $\mathbf{E}^3$. Because $\rho \circ X$ is single valued on $\Delta_0 - \{m_0\}$, we must have $\mathrm{Im}(v_{-1}) = 0$ so $v_{-1} \in \mathbf{E}^3$. If we use $(\ ,\ )$ to denote both the inner product on $\mathbf{E}^3$ and its complex extension to $\mathbf{C}^3$, we know that the conformality of $\rho \circ X$ implies $(\partial(\rho \circ X), \partial(\rho \circ X)) \equiv 0$. In particular, this gives

$$(v_{-2}, v_{-2}) = 0, \qquad (v_{-2}, v_{-1}) = 0, \qquad (v_{-1}, v_{-1}) = -2(v_{-2}, v_0).$$

It folllows that by a rotation and dilation in $\mathbf{E}^3$, we may arrange that

$$v_{-2} = \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \qquad v_{-1} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \quad (c \in \mathbf{R}).$$

We may then set $\xi = \xi^1 + i\xi^2$ and compute that

$$\rho \circ X_{|\Delta_0} = \begin{pmatrix} \xi^1|\xi|^{-2} + \mathrm{Re}(\xi h_1) \\ \xi^2|\xi|^{-2} + \mathrm{Re}(\xi h_2) \\ (c/2)\log|\xi|^2 + \mathrm{Re}(\xi h_3) \end{pmatrix} + V, \tag{4.1}$$

where h_1, h_2, and h_3 are holomorphic functions of ξ on the unit disk. We may also assume $V = 0$ by translation in $\mathbf{E}^3$. It follows that for a small disk about m_0, the image under $\rho \circ X$ looks much like an end of a catenoid. The constant $(-c)$ is called the *logarithmic growth* of the end at m_0. We are going to show that $c = 0$.

Let us suppose, contrariwise, that $c \neq 0$. Then the third component of $\partial(\rho \circ X)$ is of the form

$$(\partial(\rho \circ X)) = c\left(\xi^{-1} + a_0 + a_1\xi + a_2\xi^2 + \cdots\right) d\xi.$$

It is easy to see that there exists a unique holomorphic coordinate $\eta\colon \Delta_0 \to \mathbf{C}$ satisfying $\eta(m_0) = 0$,

$$\frac{d\eta}{d\xi}(m_0) = 1,$$

$$\frac{d\eta}{\eta} = \left(\xi^{-1} + a_0 + a_1\xi^1 + \cdots\right) d\xi.$$

In fact, we may now replace ξ by η without affecting the normalizations made thus far. This has the effect of setting $h_3 \equiv 0$ in (4.1). (Of course, if c *were* zero, we could not do this.)

Now let $I\colon \mathbf{E}^3 - \{0\} \to \mathbf{E}^3 - \{0\}$ be the inversion through the sphere of radius one centered at $0 \in \mathbf{E}^3$:

$$I\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \frac{1}{|x|^2}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

This is a conformal transformation and exchanges 0 for the "point at infinity". Because $X\colon M^2 \to S^3$ is smooth at m_0, it follows that $(I \circ \rho) \circ X\colon M \to \mathbf{E}^3$ is smooth at m_0. If we compute the third component of $(I \circ \rho) \circ X$ restricted to Δ_0 we get

$$(I \circ \rho \circ X)_3 = \frac{(c/2)|\xi|^2 \log|\xi|^2}{1 + f(\xi, \bar{\xi}) + c^2/4|\xi|^2\left(\log|\xi|^2\right)^2},$$

where f is a smooth function of ξ, $\bar{\xi}$ vanishing to second order at $\xi = 0$. It is now an elementary matter to show that, although $(\Phi \circ \rho \circ X)_3$ is always C^1 at

$\xi = 0$, it is *never* C^2 unless $c = 0$. Since $(\Phi \circ \rho \circ X)_3$ must be a *smooth* function of ξ, we see that we must have $c = 0$. This completes the proof of Theorem E.

Remark. Because the ends of $\rho \circ X\colon M^* \to \mathbf{E}^3$ are *imbedded*, a theorem of Osserman [9] asserts that

$$\int_M \Omega_X = \int_{M^*} (-K)\, dA = 2\pi(2d - \chi(M)), \tag{4.2}$$

where $d = |D| =$ the number of points in D. Now, if $(y_1) \in S^3$ is *not* in the image of X, then, taking a stereographic projection $\rho_1\colon S^3 - \{(y_1)\} \to \mathbf{E}^3$ gives an immersion $\tilde{X} = \rho_1 \circ X\colon M \to \mathbf{E}^3$ which satisfies

$$\int_M (\tilde{H}^2 - \tilde{K})\, d\tilde{A} = \int_M \Omega_X = 2\pi(2d - \chi(M)). \tag{4.3}$$

Because $\tilde{X}$ is an immersion and M is compact, the Gauss-Bonnet theorem then implies

$$\int_M \tilde{H}^2\, d\tilde{A} = 4\pi d. \tag{4.4}$$

On the other hand, Li and Yau show [8, Theorem 6] that if $\psi\colon M \to \mathbf{E}^3$ is *any* smooth immersion and k is the *maximum* number of preimages under ψ of a point in $\mathbf{E}^3$, then

$$\int_M H_\psi^2\, dA_\psi \geqslant 4\pi k. \tag{4.5}$$

In our case, $\tilde{X}^{-1}(\rho_1((y_0))) = D$ so the map $\psi = \tilde{X}$ always produces equality in their theorem.

Another remark is that a converse to Theorem E holds in the following sense: If M is an orientable surface and $X_0\colon M \to \mathbf{E}^3$ is a complete minimal immersion of finite total curvature, then the above quoted theorem of Osserman asserts that there is a compact Riemann surface $\overline{M}$, a finite set of points $D \subseteq \overline{M}$ and a diffeomorphism $M \xrightarrow{\sim} \overline{M} - D$ so that $X_0\colon \overline{M} - D \to \mathbf{E}^3$ is conformal and ∂X_0 is a meromorphic $\mathbf{C}^3$-valued 1-form on $\overline{M}$. If, moreover, the ends are imbedded and have zero logarithmic growth, then we claim that the composition $X_0\colon \overline{M} - D \to \mathbf{E}^3 \to S^3 - (y_0)$ has a smooth extension to $X\colon \overline{M} \to S^3$ with $X(D) = \{(y_0)\}$. Obviously X will be a Willmore immersion with $\mathcal{Q}_X \equiv 0$. To prove the smoothness of the extension, it suffices to note that, when an end has zero logarithmic growth (and is imbedded), then on a disk about $m_0 \in D$, we may put X_0 in the form

$$X_0 = \begin{pmatrix} \xi^1|\xi|^{-2} + \mathrm{Re}(\xi h_1) \\ \xi^2|\xi|^{-2} + \mathrm{Re}(\xi h_2) \\ \mathrm{Re}(\xi h_3) \end{pmatrix}$$

by a rotation and translation in $\mathbf{E}^3$. Here $\xi = \xi^1 + i\xi^2$ is a local holomorphic parameter with $\xi(m_0) = 0$. One then computes that

$$I \circ X_0 = \frac{1}{1 + f(\xi, \bar{\xi})} \begin{pmatrix} \xi^1 + |\xi|^2 \operatorname{Re}(\xi h_1) \\ \xi^2 + |\xi|^2 \operatorname{Re}(\xi h_2) \\ |\xi|^2 \operatorname{Re}(\xi h_3) \end{pmatrix},$$

where $f(\xi, \bar{\xi})$ is a smooth (in fact, analytic) function vanishing to second order at $\xi = 0$. Obviously $I \circ X_0$ completes smoothly across m_0.

It follows that for a compact, oriented surface M, the problem of classifying the Willmore immersion $X: M \to S^3$ satisfying $\mathcal{Q}_X \equiv 0$ is equivalent to classifying the complete minimal immersions $X^*: M^* \to \mathbf{E}^3$ of finite total curvature and with imbedded ends of zero logarithmic growth, where $M^* \simeq M - \{m_1, \cdots, m_d\}$. This latter is essentially an algebraic geometry problem. To see this, suppose we start with an X^* as above. We then give M^* the unique complex structure compatible with its orientation and the conformal structure induced by X^*. The completed surface, $\overline{M}^*$, is diffeomorphic to M by the theorem of Osserman and henceforth we identify them, writing the deleted points as $D = \{m_1, \cdots, m_d\}$. We know that

$$\partial X^* = \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix},$$

where ω^1, ω^2, ω^3 are meromorphic 1-forms on M. Our geometric data translate into holomorphic data as follows:

(i) X^* is an immersion $\Leftrightarrow$ the ω^i have no common zeros.

(ii) X^* is conformal $\Leftrightarrow$ $(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = 0$.

(iii) The ends of X^* are imbedded $\Leftrightarrow$ the ω^i have poles of at worst second order on D $\Leftrightarrow$ the ω^i are holomorphic sections of $K_M \otimes [2D]$, where K_M is the canonical bundle of M with the given complex structure.

(iv) The ends of X^* have zero logarithmic growth $\Leftrightarrow$ the ω^i are differentials of the second kind, i.e. $\operatorname{Res}_m(\omega^i) = 0$ for all i and all $m \in D$.

(v) X^* is single valued on M^* $\Leftrightarrow$ for all $\gamma \in H_1(M, \mathbf{Z})$ $\mathbf{Re}(\operatorname{Per}_\gamma(\omega^i)) = 0$ for $i = 1, 2, 3$. (Note that this is well defined because of (iv).)

Conversely, if we start with M and wish to construct the possible X^*'s, then we begin by selecting a complex structure on M and a divisor $D = m_1 + \cdots + m_d$, where the $m_\delta \in M$ are distinct. The vector space $H^0(K_M \otimes [2D])$ of holomorphic sections of $K_M \otimes [2D]$ is a complex vector space of dimension $2d + g - 1$ by the Riemann-Roch Theorem (see [5]). The subspace

$$\tilde{H}^0(K_M \otimes [2D]) = \{\omega \in H^0(K_M \otimes [2D]) | \operatorname{Res}_m(\omega) = 0 \text{ for } m \in D\}$$

is a complex vector space of dimension $d + g$. Finally, the subspace

$$V_D = \left\{\omega \in \tilde{H}^0(K_M \otimes [2D]) \,|\, \mathbf{Re}\left(\mathrm{Per}_\gamma(\omega)\right) = 0 \text{ for all } \gamma \in H_1(M, \mathbf{Z})\right\}$$

is known to be a real vector space of dimension $2d$. To construct an X^*, it remains to select three elements ω^1, ω^2, $\omega^3 \in V_D$ with no common zeros and satisfying

$$(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = 0.$$

If this can be done (it may not be possible for a given D), the required X^* is then given by the Weierstrass formula

$$X^*(M) = X^*(m_0) + \mathbf{Re}\left(\int_{m_0}^{m}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix}\right),$$

where $m_0 \notin D$. The ω^i were chosen to lie in V_D precisely to insure that the real part of the path integral in this formula should be independent of the path joining m to m_0.

In the case that $M = S^2$ we can be much more explicit for two reasons. First, S^2 has a unique conformal structure up to diffeomorphisms. It follows that there is no loss of generality in assuming that $S^2 = \mathbf{P}^1$, the complex projective line, and that the Willmore immersion under consideration is actually conformal as a map X: $\mathbf{P}^1 \to S^3$. Second, $H_1(S^2, \mathbf{Z}) = (0)$, so condition (v) above is vacuous. Indeed, we can say more: Any meromorphic 1-form ϕ on $\mathbf{P}^1$ with no residues is exact, i.e., there exists a meromorphic function f on $\mathbf{P}^1$ with $\phi = df$. This is more or less clear, but for a proof, see [5].

This brings us to our main theorem:

Theorem F. *Let X: $\mathbf{P}^1 \to S^3$ be a conformal Willmore immersion. There exist a point $(y_0) \in X(\mathbf{P}^1)$ (unique if X is not totally umbilic) so that $D = X^{-1}((y_0))$ is a divisor in $\mathbf{P}^1$ with distinct points, a stereographic projection ρ: $S^3 - \{(y_0)\} \to \mathbf{E}^3$ and a meromorphic curve f: $\mathbf{P}^1 \to \mathbf{C}^3$ with simple poles along D so that $\rho \circ X = \mathbf{Re}(f)$.*

Moreover f is an immersion with null tangents (i.e. $(df, df) \equiv 0$). Conversely, if f: $\mathbf{P}^1 \to \mathbf{C}^3$ is a meromorphic immersion with simple poles along D and null tangents, then $\mathbf{Re}(f)$: $\mathbf{P}^1 - D \to \mathbf{E}^3$ *completes smoothly across D to be a conformal Willmore immersion* $(\mathbf{Re}(f))$: $\mathbf{P}^1 \to S^3$.

Proof. By Theorem D, $\mathcal{Q}_X \equiv 0$. If X is all umbilic, choose (y_0) arbitrarily on $X(\mathbf{P}^1) \subseteq S^3$. Then any stereographic projection ρ: $S^3 - \{y_0\} \to \mathbf{E}^3$ makes $\rho \circ X$: $\mathbf{P}^1 - D \to \mathbf{E}^3$ a stereographic projection onto a plane. Note that in this case D is a single point. If X is not all umbilic, we apply Theorem E. In either case, $\rho \circ X$: $\mathbf{P}^1 - D \to \mathbf{E}^3$ is a conformal minimal immersion which is complete, has finite total curvarture and imbedded ends with zero logarithmic growth. It follows that $\partial(\rho \circ X)$ is a meromorphic $\mathbf{C}^3$-values 1-form with no residues and double poles along D. By our discussion above, there exists a meromorphic f: $\mathbf{P}^1 \to \mathbf{C}^3$ so that $df = 2\partial(\rho \circ X)$ and f clearly must have simple poles along D. Now

$$d(\rho \circ X) = \partial(\rho \circ X) + \bar{\partial}(\rho \circ X) = \tfrac{1}{2}(df + d\bar{f}) = d\mathbf{Re}(f).$$

By adding a constant to f we may arrange that $\rho \circ X = \mathbf{Re}(f)$. Note that f is an immersion since $\rho \circ X$ is and that

$$(df, df) = 4(\partial(\rho \circ X), \partial(\rho \circ X)) \equiv 0.$$

The converse is now elementary. q.e.d.

The determination of the meromorphic null curves f: $M \to \mathbf{C}^3$ (where M is a Riemann surface) is classical, see [4]. For such an f, either $f(M)$ is a null line in $\mathbf{C}^3$ or else there exist meromorphic functions g, h on M with g nonconstant so that

$$f = \begin{pmatrix} i/2 & -i/2g & i(1 + g^2/4) \\ -\frac{1}{2} & \frac{1}{2}g & (1 - g^2/4) \\ 0 & -1 & g \end{pmatrix} \begin{pmatrix} h \\ h' \\ h'' \end{pmatrix},$$

where h' and h'' are meromorphic functions on M defined by $h' = dh/dg$ and $h'' = dh'/dg$.

We then have

$$df = \begin{pmatrix} i(1 + g^2/4) \\ (1 - g^2/4) \\ g \end{pmatrix} dh''.$$

If we regard g as a holomorphic map g: $M \to \mathbf{P}^1$, then g is the Gauss map of the minimal immersion $\mathbf{Re}(f)$: $M - D \to \mathbf{E}^3$ (D is the polar divisor of f), see [9].

Unfortunately, it appears to be a nontrivial algebraic problem, even when $M = \mathbf{P}^1$, to specify g and h so that the resulting f will be a meromorphic immersion with simple poles.

5. An example and further results

It follows from our results in §4 that the critical values of the modified Willmore functional

$$\mathscr{W}_{S^2}(X) = \int_{S^2} \Omega_X$$

for immersion X: $S^2 \to S^3$ lie in the discrete set $\{4\pi d | d \geqslant 0\}$. Obviously, if $\mathscr{W}_{S^2}(X) = 0$, then $\Omega_X \equiv 0$ so that X is totally umbilic and hence X gives a diffeomorphism of S^2 onto a round 2-sphere in S^3. Thus, all these Willmore immersions are equivalent modulo reparametrizations in S^2 and conformal transformations in S^3.

If X: $S^2 \to \mathbf{S}^3$ is a Willmore immersion with $\mathscr{W}_{S^2}(X) = 4\pi d > 0$, then X is not totally umbilic and the associated meromorphic null curve f: $\mathbf{P}^1 - D \to \mathbf{C}^3$ has $d + 1$ poles. It is easy to see that the requirement that f be an *immersion* with simple poles eliminates the possibilities $d = 1, 2$. Thus 4π and 8π are not critical values of $\mathscr{W}_{S^2}$. When $d = 3$, a calculation shows that there is a meromorphic coordinate z: $\mathbf{P}^1 - (p_\infty) \to \mathbf{C}$ so that D is given by

$$D = \{p_\infty\} \cup \{p \in \mathbf{P}^1 | (z(p))^3 = 1\}.$$

In fact, if we let ε denote a nontrivial cube root of unity, then the curve

$$f = v_0 z - \frac{v_1}{z - \varepsilon} - \frac{v_2}{z - \varepsilon^2} - \frac{v_3}{z - 1} + f_0,$$

where $f_0 \in \mathbf{C}^3$ and $v_0, v_1, v_2, v_3 \in \mathbf{C}^3$ satisfy

$$(v_i, v_j) = \lambda \neq 0, \qquad 1 \leqslant i < j \leqslant 3,$$
$$(v_i, v_i) = 0, \qquad 1 \leqslant i \leqslant 3,$$
$$v_0 = \tfrac{1}{3}(\varepsilon v_1 + \varepsilon^2 v_2 + v_3),$$

is the most general meromorphic null immersion with polar divisor D.

Now it is easy to see that two such curves, f and $\tilde{f}$, determine conformally equivalent map, $(\mathbf{Re}(f))$: $\mathbf{P}^1 \to S^3$ and $(\mathbf{Re}(\tilde{f}))$: $\mathbf{P}^1 \to S^3$ if and only if the minimal immersions $\mathbf{Re}(f)$ and $\mathbf{Re}(\tilde{f})$: $\mathbf{P}^1 - D \to \mathbf{E}^3$ differ by Euclidean motions and dilations. By translation and dilation in $\mathbf{E}^3$ we may normalize our maps f and $\tilde{f}$ so that $f_0 = \tilde{f}_0 = 0$ and $|\lambda| = |\tilde{\lambda}| = 1$.

Now every real rotation R: $\mathbf{E}^3 \to \mathbf{E}^3$ extends complex linearly to R: $\mathbf{C}^3 \to \mathbf{C}^3$. This imbeds SO(3, $\mathbf{R}$) into SO(3, $\mathbf{C}$). Clearly, there exists an $A \in$ SO(3, $\mathbf{C}$) and a $\mu \in \mathbf{C}$ satisfying

$$\tilde{v}_i = \mu A v_i, \qquad 0 \leqslant i \leqslant 3$$

(we have $\tilde{\lambda} = \mu^2 \lambda$). It is now not difficult to show that $\mathbf{Re}(f)$ and $\mathbf{Re}(\tilde{f})$ differ by a Euclidean motion if and only if $A \in$ SO(3, $\mathbf{R}$) and $\mu = 1$.

It follows that, after taking into account reparametrization in S^2 and conformal transformations in S^3, the moduli space for Willmore immersions with $d = 3$ (i.e. $\mathscr{W}_{S^2}(X) = 12\pi$) is SO(3, $\mathbf{C}$)/SO(3, $\mathbf{R}$) $\times S^1$ modulo the action of A_4, the alternating group on four letters (permuting the points of D by linear fractional transformations on $\mathbf{P}^1$). Surprisingly, this space is *not* compact and is of dimension 4 (at its smooth points).

Now a similar situation holds for $d = 2n + 1$, $n \geqslant 1$. One can show that, modulo the obvious equivalences, the moduli space of Willmore immersions X: $S^2 \to S^3$ with $\mathscr{W}_{S^2}(X) = 4\pi d$ is nonempty and of dimension $4n$ at its smooth points.

For even values of d, this author does not know whether the moduli space is nonempty.

In view of Theorem C, it seems natural to attempt to extend the Willmore functional to the space of conformal branched immersions X: $M^2 \to S^3$, where M is an oriented surface with a fixed conformal structure. In fact this can be done, though considerable care must be exercised in extending the conformal Gauss map γ_X: $M^2 \to Q$ across the branching divisor B. Once this is done, Ω_X is again seen to be a smooth 1-form on X. The condition $\delta\Omega_X = 0$ then allows us to construct $\hat{X}$: $M^2 \to S^3$ in a smooth manner. It, too, is a branched conformal Willmore immersion. In fact, if we let $\hat{B}$ denote the branching divisor of $\hat{X}$ and let U_+ denote the divisor

$$U_+ = \sum_{m \in M} k(m)m \geqslant 0,$$

then, for M compact and connected, either

(i) X is totally umbilic,

(ii) $\hat{X}$ is constant, or

(iii) we have the equation of line bundles

$$K_M = [B] \otimes [\hat{B}] \otimes [2U_+].$$

Thus, for $M = S^2$, the third possibility cannot occur.

When M is a torus, K_M has degree zero so either X: $M^2 \to S^3$ is a branched cover of a round 2-sphere, comes from a minimal surface in $\mathbf{E}^3$ by stereographic projection, or else X and $\hat{X}$ are both imersions ($B = \hat{B} = 0$) and $U_+ = 0$. In the first two cases, $\mathscr{W}_M(X) = 4\pi d$, where $d \geqslant 2$. In the third case, the Clifford torus in S^3 furnishes a Willmore surface with $\mathscr{W}_M(X) = 2\pi^2 < 8\pi$ (see [12]).

Whenever the second possibility occurs, a stereographic projection ρ: $S^3 - \{\hat{X}(M)\} \to \mathbf{E}^3$ makes $\rho \circ X$: $M - D \to \mathbf{E}^3$ a complete branched minimal immersion of finite total curvature (here $D = \hat{X}^{-1}(\hat{X}(m))$). The meromorphic form $\partial(\rho \circ X)$ still has poles along D but they may be of higher order than 2. It seems likely that $\partial(\rho \circ X)$ has no residues, but we have not proved this.

References

[1] É. Cartan, *Les espaces à connexion conforme*, Ouevres Complete, III Vol. 1, 747–797.

[2] ———, *Théorie des groupes fini et continus et la géométrie différentielle traitées par la méthode du repère mobile*, Gauthier-Villars, Paris, 1937.

[3] S. S. Chern, P. Hartman & A. Winter, *On isothermic coordinates*, Comment Math. Helv. **28** (1954).

[4] G. Darboux, *Leçons sur la théorie générale des surfaces*, Chelsea, New York.

[5] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, New York 1978.

[6] R. Gulliver, R. Osserman & H. Royden, *A theory of branched immersions*, Amer. J. Math. **95** (1973) 750–812.

[7] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1982.

[8] P. Li & S. T. Yau, *A new conformal invariant and its aplications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982) 269–291.

[9] R. Osserman, *Minimal surfaces, Gauss maps, total curvature, eigenvalue estimates, and stability*, The Chern Symposium, Springer, New York, 1979.

[10] R. O. Wells, Jr., *Differential analysis on complex manifolds*, Springer, New York, 1973.

[11] J. H. White, *A global invariant of conformal mappings in space*, Proc. Amer. Math. Soc. **88** (1973) 162–164.

[12] T. J. Willmore, *Note on embedded surfaces*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Seçt. I a Mat. **11** (1965) 493–496.

RICE UNIVERSITY